Journal of Statistical Physics, Vol. 51, Nos. 5/6, 1988

Front Propagation in Certain One-Dimensional Exclusion Models

Maury Bramson[1]

Received February 24, 1988

Recent results on two interacting particle systems on $\mathbb{Z}$ are summarized, the asymmetric simple exclusion process and the branching exclusion process.

KEY WORDS: Exclusion process; branching exclusion process; Burgers equation; shocks; hydrodynamic limit; KPP equation.

1. INTRODUCTION

In this paper I consider two classes of interacting particle systems on the integers. They are briefly described as follows. At any time t particles are present at sites in $\mathbb{Z}$, with at most one particle per site. After exponentially distributed random times with mean 1, particles attempt to execute simple random walks, with a particle at x attempting to jump to $x+1$ with probability p and to $x-1$ with probability $q=1-p$. This jump occurs if the target site is unoccupied; otherwise, the particle remains at x until the next attempt, for which the "exponential clock" starts over again. We are assuming that all these random times are independent of one another. Thus, no two attempted jumps occur simultaneously. The assumption that times are exponentially distributed implies the memory-less property. Such a process is therefore Markov, with the future after time t not depending on behavior at times $s<t$ if the state at t is already known.

This class of interacting particle systems is called the exclusion process. We will assume the initial state $(t=0)$ is given by appropriate combinations of the product measures ν_α, $0 \leqslant \alpha \leqslant 1$. The measure ν_α is defined so that at each site of $\mathbb{Z}$ there is a particle with probability α and these events are independent. We denote the process by η_t. As the behavior of η_t

[1] Mathematics Department, University of Wisconsin, Madison, Wisconsin.

0022-4715/88/0600-0863$06.00/0

for $p=q=1/2$ is well understood, we restrict our attention here to the asymmetric case with $p>1/2$. (The behavior for $p<1/2$ is of course analogous.) A second class of particle systems arises if one also allows branching. As before, the interaction is described by exponential times involving occupied sites x and neighbors $x \pm 1$. If the target site $x+1$ (or similarly, $x-1$) is unoccupied, then the particle at x, while remaining at x, gives birth to a particle at $x+1$ ($x-1$). We will assume that the initial state consists of particles at all sites $x \leqslant 0$ and no particles at $x>0$. For this class of particle systems, the branching exclusion process, the exponential random times associated with the random walk are assumed to occur at rate $\beta>0$ and the exponential random times with branching at rate 1. We also restrict ourselves to the case where $p=q=1/2$ for both the random walk and branching mechanisms. We denote this process by ${}_\beta\xi_t$. A general reference for the exclusion process is ref. 1. The material covered here on the exclusion process is from refs. 2–7; its content is similar to that of ref. 6. The material on the branching exclusion process is from refs. 8 and 9.

2. EXCLUSION PROCESS

The particle system ζ_t is said to be in equilibrium if ζ_t has the same distribution for all t. A distribution is translation-invariant if it remains the same under the translation $x \mapsto x+1$. It can be checked that ν_α are equilibrium probability measures for the exclusion process η_t. It is well known that they are the only such translation-invariant probability measures. It is natural to ask what behavior ensues as $t \to \infty$ if η_t is started at a nonequilibrium state.

Perhaps the most obvious choice of a nonequilibrium state is

$$\begin{aligned} \eta_0 &= \nu_\rho \qquad \text{on} \quad 0, 1, 2,... \\ &= \nu_\lambda \qquad \text{on} \quad -1, -2,... \end{aligned}$$

where $0 \leqslant \rho, \lambda \leqslant 1$. On the positive integers η_0 has density ρ and on the negative integers density λ. Presumably some manner of mixing of densities occurs for η_t as $t \to \infty$. This is indeed the case for $p=1/2$, where in fact[1]

$$\eta_t \xrightarrow{\text{w}} \nu_{(\rho+\lambda)/2} \qquad \text{as} \quad t \to \infty \tag{1}$$

Here $\to^{\text{w}}$ means that over any finite subset A of $\mathbb{Z}$, the restrictions of the distributions of η_t and $\nu_{(\rho+\lambda)/2}$ to A become indistinguishable as $t \to \infty$. (This is weak convergence in the appropriate topology.)

In our setting with $p>1/2$, the behavior of η_t turns out to be more complicated, as evidenced by the following result.[2]

Theorem 1:

(a) If $\lambda \geqslant 1/2$ and $\rho \leqslant 1/2$, then $\eta_t \to^{w} \nu_{1/2}$.
(b) If $\rho \geqslant 1/2$ and $\lambda + \rho > 1$, then $\eta_t \to^{w} \nu_\rho$.
(c) If $\lambda \leqslant 1/2$ and $\lambda + \rho < 1$, then $\eta_t \to^{w} \nu_\lambda$.

Theorem 1 partitions the unit square $\{(\lambda, \rho): 0 \leqslant \lambda, \rho \leqslant 1\}$ into three basic regions: (a) the square of length 1/2 in the lower right-hand corner, where the limit is $\nu_{1/2}$, and the regions (b) and (c) obtained by dividing the remaining area by the line segment between (0, 1) and (1/2, 1/2). One can show by substituting occupied sites for unoccupied sites and unoccupied sites for occupied sites, and by interchanging p and q, that the behavior given in (b) must imply that in (c), and vice versa. It is not easy to see, however, how the limiting densities 1/2, ρ, and λ come about, and why these particular densities hold for the given ranges. Also note that no assertion is made in Theorem 1 regarding the asymptotic behavior for η_t if $\lambda + \rho = 1$ and $0 \leqslant \lambda < 1/2$ [the line segment between (0, 1) and (1/2, 1/2)].

A much clearer picture of the reasons for the behavior in (a)–(c) was given in refs. 3 and 4. Instead of considering the asymptotic behavior of η_t in only the stationary coordinate system, one can translate at any fixed rate, and then examine η_t under these linear translations. Presumably one should obtain as the limit ν_α for appropriate choice of α. In fact, one obtains:

Theorem 2. Suppose that either $\lambda \geqslant \rho$ or that $x \neq (p-q) \cdot (1-\lambda-\rho)t$. Then,

$$\eta_{t/\varepsilon}(\cdot + x/\varepsilon) \xrightarrow{w} \nu_{u(t,x)} \quad \text{as} \quad \varepsilon \to 0 \tag{2}$$

where $u(t, x)$ satisfies

$$u_t + (p-q)[u(1-u)]_x = 0 \tag{3}$$

with

$$\begin{aligned} u(0, x) &= \rho \quad \text{if} \quad x \geqslant 0 \\ &= \lambda \quad \text{if} \quad x < 0 \end{aligned} \tag{4}$$

The limit in (2) states that if one is moving at rate x/t, then one sees in the limit a product measure with density $u(t, x)$. The function $u(t, x)$ will be the solution to the Burgers equation given in (3)–(4) (with the usual entropy conditions). For $\lambda < \rho$, one needs to avoid the shock at $x = (p-q) \cdot (1-\lambda-\rho)$. Rescaling such as in (2) has been applied increasingly frequently in the literature to a wide range of processes. The basic idea is that (2) is a "hydrodynamic" (macroscopic) limit of a microscopic model η with certain dynamics. Under this rescaling, one obtains a limiting density u, which is to be thought of as representing some macroscopically observable quantity. In our case, u is of course the density of particles.

It is not difficult to solve (3)–(4) using the method of characteristics. The special case with $x=0$ will imply (a)–(c) of Theorem 1. Break the problem into case I, with $\lambda \geqslant \rho$, and case II, with $\lambda < \rho$. In case I, the solution $u(t, x)$ has no shocks because the characteristics $x_1 = (p-q) \cdot (1-2\rho)t$ and $x_2 = (p-q)(1-2\lambda)t$ move apart with $x_1 > x_2$. Therefore, $\mathbb{R}$ can be divided into three intervals with

$$
u(t,x) = \begin{array}{ll} \rho & \text{for} \quad x \geqslant (p-q)(1-2\rho)t \\ 1/2 - x/2t & \text{for} \quad (p-q)(1-2\lambda)t \leqslant x \leqslant (p-q)(1-2\rho)t \\ \lambda & \text{for} \quad x \leqslant (p-q)(1-2\lambda)t \end{array} \tag{5}
$$

(the middle interval is linear in x). Note in particular that under scenario (a), $x=0$ is contained in the middle interval, whereas under (b) and (c), $x=0$ lies in the right, resp. left interval. Plugging $x=0$ into (5), one obtains Theorem 1 under $\lambda \geqslant \rho$. In case II, one obtains the solution

$$
\begin{aligned} u(t,x) &= \rho \qquad \text{for} \quad x > (p-q)(1-\lambda-\rho)t \\ &= \lambda \qquad \text{for} \quad x < (p-q)(1-\lambda-\rho)t \end{aligned} \tag{6}
$$

with a discontinuity at $x = (p-q)(1-\lambda-\rho)t$. Under the two permissible scenarios (b) and (c) from Theorem 1, $x=0$ is contained in the right, resp. left interval. Theorem 1 with $\lambda < \rho$ therefore follows from (6). The proof of Theorem 2 itself of course requires work. It is not difficult to show, however, that once one knows (2), then (3)–(4) must follow.

Returning to Theorem 1, we recall the case $\lambda + \rho = 1$ with $0 \leqslant \lambda < 1/2$, for which no assertion was made. The set lies on the boundaries of both regions (b) and (c), which have corresponding densities ρ and λ. It is therefore plausible that something unstable is occurring along this interface. Further support is provided by Theorem 2, which gives $x=0$ as the site of the shock between $u=\rho$ and $u=\lambda$. In fact, the following holds[5]:

Theorem 3. Suppose that $\lambda + \rho = 1$ and $0 < \lambda < 1/2$. Then,

$$
\eta_t \xrightarrow{\mathrm{w}} \tfrac{1}{2}\nu_\rho + \tfrac{1}{2}\nu_\lambda \qquad \text{as} \quad t \to \infty \tag{7}
$$

In other words, Theorem 3 states that over any interval $[-M, M] \subset \mathbb{Z}$, if t is large enough, then η_t resembles a mixture of ν_ρ and ν_λ. In particular, unlike the case for regions (a)–(c), $\lim_{t \to \infty} \eta_t$ is not ergodic. Note that the sole remaining case, where $\rho = 1$ and $\lambda = 0$, is easy to treat. Since $p > 1/2$, particles tend to drift to the right, and so it will be the case that except for a probability that is decreasing exponentially in x, all points to the right of x will be occupied and all points to the left of $-x$ unoccupied at a given t.

The result in Theorem 3 brings to mind extensions in two possible directions. Presumably, the behavior at the shocks discussed in Theorem 2 that run along $x = (p-q)(1-\lambda-\rho)t$ with $\lambda < \rho$ and $\lambda + \rho \neq 1$ should be analogous to that given in (7) for $\lambda + \rho = 1$. This has not yet been demonstrated. The proof of Theorem 3 uses reflection symmetries, which are affected by translation. One can also attempt to analyze the asymptotic behavior "at a shock" in more detail, without rescaling space. In this setting, is the shock still sharp ("of bounded width"), and where is it located? These questions are answered in ref. 7 in the special case where $\lambda = 0$. One does indeed have a sharp shock, and its location is given by Brownian motion. The assumption $\lambda = 0$ implies the existence of a leftmost particle at all times, with respect to which one can center and analyze the process.

3. BRANCHING EXCLUSION PROCESS

Start the branching exclusion process ${}_\beta\xi_t$ with the initial state

$$\begin{aligned} {}_\beta\xi_0(x) &= 0 \qquad \text{for} \quad x \geqslant 0 \\ &= 1 \qquad \text{for} \quad x < 0 \end{aligned} \tag{8}$$

The rightmost particle ${}_\beta X_t$ is then well-defined for all t. We will be interested in examining the asymptotic behavior of ${}_\beta X_t$. [Since we are assuming that $p = q = 1/2$, the bottom half of (8) could be omitted as long as there is a particle somewhere.]

One can show that the number of empty sites to the left of ${}_\beta X_t$ remains bounded as $t \to \infty$. (More precisely, the configuration as seen from ${}_\beta X_t$ defines a positive recurrent Markov process.) It is therefore not difficult to show that the asymptotic velocity of ${}_\beta X_t$,

$$V(\beta) \equiv \lim_{t \to \infty} E[{}_\beta X_t]/t \tag{9}$$

exists for all $\beta > 0$. One can, if one wishes, omit the expected value $E[\cdot]$ in (9), since the pathwise limits of ${}_\beta X_t/t$ also exist. Clearly, $V(\beta) \to \infty$ as $\beta \to \infty$; we wish to know at what rate. Since random walk has variance β, $\sqrt{\beta}$ will be the right scaling. What is $\lim_{\beta \to \infty} V(\beta)/\sqrt{\beta}$?

To provide motivation, we recall the following result from ref. 8.

Theorem 4. As $\beta \to \infty$,

$$P[[\beta^{1/2}x] \in {}_\beta\xi_t] \to u(t, x) \tag{10}$$

where $u(t, x)$ satisfies

$$u_t = \tfrac{1}{2}u_{xx} + u(1-u) \tag{11}$$

with

$$\begin{aligned} u(0, x) &= 0 \quad \text{for} \quad x \geqslant 0 \\ &= 1 \quad \text{for} \quad x < 0 \end{aligned} \tag{12}$$

($[y]$ denotes the integer part of y.)

The limit in (10) is in some ways analogous to that in (2) of Theorem 2. Space is scaled differently, however, and, as mentioned at the beginning of the section, occupation of different sites is strongly correlated, unlike the independence in (2). The macroscopic density $u(t, x)$ satisfies the KPP (Kolmogorov–Petrovsky–Piscounov) equation (11) with initial data corresponding to (8). It is well known that the solution approaches a traveling wave $w(x)$ moving at rate $\sqrt{2}$:

$$u(t, x+m(t)) \to w(x) \quad \text{uniformly in } x \text{ as } t \to \infty \tag{13}$$

with

$$dm(t)/dt \to \sqrt{2} \qquad \text{as} \quad t \to \infty \tag{14}$$

One has $w(x) \to 0$ as $x \to \infty$, and $w(x) \to 1$ as $x \to -\infty$.

Returning to $V(\beta)$, it is now not difficult to guess the correct limit from Theorem 4 and (13)–(14) by inverting limits. In ref. 9 the following is demonstrated:

Theorem 5. $V(\beta)/\sqrt{\beta} \to \sqrt{2}$ as $\beta \to \infty$.

One can ask about various possible extensions of Theorem 5. For instance, in the spirit of Theorem 4, how does ξ_t approximate $u(t, x)$ as both $t, \beta \to \infty$? One can also generalize the random walk and branching rules for ${}_\beta\xi_t$, so that they are finite range, but not necessarily nearest neighbor, and show the appropriate analog of Theorem 5.

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